LECTURE 9

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1. Geometric properties

We introduce certain geometric properties of affine algebraic varieties. The word "geometric" can be understood as indicating that these properties only depends on $\bar{k}$, not on k.

1.1. Dimension.

Definition 1.2. Let $V \subset \mathbb{A}^n_k$ be an affine algebraic variety. Recall we have an associated field $\bar{K}(V) := \mathrm{Frac}(\bar{K}[X]/I(V))$. Define the dimension of V to be the transcendence degree of $\bar{K}(V)$ over $\bar{K}$.

Definition 1.3. $\dim \mathbb{A}^n_k = n$. More generally, if $V \subset \mathbb{A}^n_k$ is a d-dimensional affine subspace (i.e. a translate of a d-dimensional sub-vector space of $\bar{k}^n$), then $\dim V = d$.

Remark 1.4. For a general algebraic set in $\mathbb{A}^n_k$, its irreducible components can have different dimensions. For example, $V = V(XZ, YZ) \subset \mathbb{A}^3_k$ is an algebraic set. It has two irreducible components: the z-axis and the $x - y$ plane.

Theorem 1.5. *Let $V \subset \mathbb{A}^n_k$ be an affine algebraic variety. Then $\dim V = n - 1$ if and only if there is a nonzero irreducible polynomial $f(X) \in \bar{k}[X]$ such that $V = V(f)$. Such a V is called a hypersurface.*

Example 1.6. $V(X^n + Y^n - 1) \subset \mathbb{A}^2_{\mathbb{Q}}$ is a one-dimensional variety.

1.7. **The local rings.** Let $V \subset \mathbb{A}^n_k$ be an affine algebraic variety. Let $p \in V$. Recall that p determines a maximal ideal $\mathfrak{m}_p \subset \bar{k}[V]$, defined by $\mathfrak{m}_p := \left\{f \in \bar{k}[V] | f(p) = 0\right\}$. Define the local ring of V at p, denoted by $\mathcal{O}_{V,p}$, to be the localization of $\bar{k}[V]$ at $\mathfrak{m}_p$ i.e. the subring $\left\{f/g \in \bar{k}(V) | f, g \in \bar{k}[V],\ g(p) \neq 0\right\}$ of $\bar{k}(V)$. We will still use $\mathfrak{m}_p$ to denote the unique maximal ideal of $\mathcal{O}_{V,p}$. Note that we have $\mathcal{O}_{p,V}/\mathfrak{m}_p \xrightarrow{\sim} \bar{k}$, the identification given by evaluating functions in $\mathcal{O}_{V,p}$ at p. The quotient $\mathfrak{m}_p/{\mathfrak{m}_p}^2$ is naturally a module over $\mathcal{O}_p/\mathfrak{m}_p \cong \bar{k}$. Define the dimension of V at p to be $\dim_{\bar{k}} \mathfrak{m}_p/{\mathfrak{m}_p}^2$.

1.8. Smoothness.

Definition 1.9. Let $V \subset \mathbb{A}^n_k$ be an affine algebraic variety. Let $p \in V$. We say V is smooth, or nonsingular, at p, if there exists a set of generators $f_1, \cdots, f_m$ of $I(V)$, the matrix $(\frac{\partial f_i}{\partial X_j}(P))_{1 \leq i, \leq m, 1 \leq j \leq n}$ has rank equal to $n - \dim V$. If V is nonsingular at any point in V, then we say V is nonsingular.

Proposition 1.10. *Smoothness is independent of the choice of the set of generators of $I(V)$. Moreover, V is nonsingular at $p \in V$ if and only if $\dim_{\bar{k}} \mathfrak{m}_p/{\mathfrak{m}_p}^2 = \dim V$.*

Example 1.11. Let $V = V(f) \subset \mathbb{A}_k^n$ be a hypersurface. Then V is singular at a point $p \in V$ if and only if $\partial f / \partial X_i(P) = 0, \forall 1 \le i \le n$.

Example 1.12. Suppose $\operatorname{char} k \neq 2, 3$. Let $V_1 = V(Y^2 - X^3 - X), V_2 = V(Y^2 - X^3 - X^2) \subset \mathbb{A}_{\bar{k}}^2$. V_1 is nonsingular. V_2 is singular at the point $(0,0)$.

2. Projective varieties

Roughly speaking, projective varieties are zero loci of homogeneous polynomials in projective spaces. Why do we introduce projective spaces? Consider for example the question of finding the intersection points of a line and a conic in the plane. Since they are defined by equations of degrees 1 and 2 respectively, one expects to find $2 = 1 \cdot 2$ intersection points in general.

In Cartesian geometry, there are three exceptions. Firstly, we may have a circle $X^2 + Y^2 = 1$ and a line $X = 2$, that do not intersect over $\mathbb{R}$. This is resolved by looking at solutions in $\mathbb{C}$, and in this case we have two solutions $(2, \pm\sqrt{-3})$.

Secondly, the line in question may be tangent to the conic. In this case we only have one intersection point, even over $\mathbb{C}$. However there is a precise sense in which a tangent point should count as an intersection of multiplicity 2, and if we take multiplicities into account, we are still in good shape.

The third issue is that we may have a line $X = Y$ and a hyperbola $X^2 - Y^2 = 1$, which are asymptotic to each other. In this situation we would like to think that they are tangent to each other "at infinity", i.e. we have an intersection "at infinity" of multiplicity 2. The projective spaces are precisely the extension of the ordinary affine spaces that include these points at infinity.

We fix a perfect field k with algebraic closure $\bar{k}$.

Definition 2.1. The n-dimensional projective space over k is $\mathbb{P}_k^n := \bar{k}^{n+1} - \{0\} / \sim$. Where $(x_0, \cdots, x_n) \sim (y_0, \cdots, y_n)$ if $\exists \lambda \in \bar{k}^\times$ such that $y_i = \lambda x_i, \forall i$. The point represented by $(x_0, \cdots, x_n)$ is denoted by $[x_0 : \cdots : x_n]$. This notation is called the homogeneous coordinates. The Galois group G_k acts on $\mathbb{P}_k^n$. Let $k \subset l \subset \bar{k}$. Define $\mathbb{P}_k^n(l) := \{[x_0 : \cdots : x_n] | x_i \in l, \forall i\}$. We have $\mathbb{P}_k^n(l) = (\mathbb{P}_k^n)^{\mathrm{Gal}(\bar{k}/l)}$.

Remark 2.2. If $[x_0 : \cdots : x_n] \in \mathbb{P}_k^n(l)$, it does not necessarily follow that each $x_i \in l$. However, for any i such that $x_i \neq 0$, we have $x_j / x_i \in l, \forall j$.

Let f be a polynomial in $\bar{k}[X_0, \cdots, X_n]$. Then the value of f at a point $p \in \mathbb{P}_k^n$ is not well defined. However, if f is homogeneous, then the condition $f(p) = 0$ has a well defined meaning. Thus if $\{f_i\}$ is a set of homogeneous polynomials, we can define its common zero locus $V(\{f_i\}) \subset \mathbb{P}_k^n$. We observe that if $I \subset \bar{k}X_0, \cdots, X_n$ is the ideal generated by $\{f_i\}$, then for any $g \in I$ that is also homogeneous, g vanishes on $V(\{f_i\})$.

Definition 2.3. An ideal $I \subset \bar{k}[X_0, \cdots X_n]$ is said to be homogeneous, if it is generated by homogeneous polynomials.

Definition 2.4. Let $I \subset \bar{k}[X_0, \cdots X_n]$ be a homogeneous ideal. Define its zero locus $V(I) \subset \mathbb{P}_k^n$ to be the set $\{[X_0 : \cdots : X_n] | f(X_0, \cdots, X_n) = 0, \forall \text{ homogeneous } f \in I\}$. A subset of $\mathbb{P}_k^n$ of this form is called a projective algebraic set.

Definition 2.5. Let $V \subset \mathbb{P}_k^n$ be a projective algebraic set. Define the homogeneous ideal of V to be the ideal $I(V)$ of $\bar{k}[X_0, \cdots, X_n]$ generated by the homogeneous polynomials that vanish on V. If $I(V)$ is generated by homogeneous polynomials in $k[X_0, \cdots X_n]$, we say V is defined over k. In this case G_k acts on V, and for $k \subset l \subset \bar{k}$, the set $V(l) := V \cap \mathbb{P}^n_k(l)$ is the set of points in V fixed by $\mathrm{Gal}(\bar{k}/l)$.

Example 2.6. Let $V = V(X^2 + Y^2 - 3Z^2) \subset \mathbb{P}^2_{\mathbb{Q}}$. Then V is defined over $\mathbb{Q}$, and $V(\mathbb{Q}) = \emptyset$. To see this, suppose $p = [x : y : z] \in V(\mathbb{Q})$. We may assume $xy, z \in \mathbb{Z}$ and $\gcd(x, y, z) = 1$. Then $x^2 + y^2 \equiv 0 \mod 3$. Since -1 is not a square modulo 3, we have $x \equiv y \equiv 0 \mod 3$. But then $9 | x^2 + y^2 = 3z^2$, so $3|z$, contradiction.

Definition 2.7. Let $V \subset \mathbb{P}^n_k$ be a projective algebraic set. We say V is a projective variety, if $I(V)$ is a prime ideal of $\bar{k}[X_0, \cdots, X_n]$.

Remark 2.8. We used the associated ideal / homogeneous ideal to an affine / projective algebraic set to define whether it is a variety. However there is an intrinsic definition using the Zariski topology, so that an affine / projecrive algebraic set is a variety if and only if it is irreducible under the Zariski topology. As in the affine case, any projective algebraic set can be uniquely written as a union of its irreducible components, which are projective varieties.